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1:2 INTERNAL RESONANCE OF COUPLED DYNAMIC SYSTEM WITH QUADRATIC AND CUBIC NONLINEARITIES *

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Abstract: *The* 1:2 *internal resonance of coupled dynamic system with quadratic and cubic nonlinearities is studied. The normal forms of this system in* 1:2 *internal resonance were derived by using the direct method of normal form. In the normal forms, quadratic and cubic nonlinearities were remained. Based on a new convenient transformation technique, the* 4*-dimension bifurcation equations were reduced to* 3*-dimension. A bifurcation equation with one-dimension was obtained. Then the bifurcation behaviors of a universal unfolding were studied by using the singularity theory. The method of this paper can be applied to analyze the bifurcation behavior in strong internal resonance on* 4*-dimension center manifolds.*

Key words: quadratic and cubic nonlinearities; Normal Form; 1:2 internal resonance
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Introduction

From the beginning of 1990s, research on system with engineering practical background becomes more and more attractive by using the nonlinear dynamic system theory.

As a method of simplifying differential equations, the theory of Normal Form gives the simplest forms of differential equations in the neighborhood of an equilibrium point or a periodic orbit; it makes the research on bifurcation simpler. The proof and analysis on the results are simplified as well.

Recently a large amount of research has been related to nonlinear systems having multi-degree of freedom in strong internal resonance. A.H.Nayfeh[1] studied the 1:2 internal resonance system with quadratic nonlinearities; W.F.Langford[2] researched the 1:1 internal resonance of system with quadratic and cubic nonlinearities; W.F.Langford[3] studied the Normal Form of 1:2

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Biography: CHEN Yu-shu (1931 –), Professor

internal resonance only up to order two; WU Zhi-qiang[4] derived the expression of the coefficients of the linear unfolding term and up through terms of order three, as well as the formulas for finding the coefficients of higher-order terms in Normal Form for non-semisimple 1:1 internal resonance bifurcation, and obtained the formulas to calculate the Normal Form of multiple non-resonance Hopf bifurcation. CHEN Fang-qi[5] studied the universal unfolding on a bifurcation equation of a parameter excited system having one-degree of freedom. But any of this can hardly be found in the many existing papers related to the coupled systems having quadratic and cubic nonlinearities at the same time in 1:2 internal resonance. So it is very necessary to study this kind of situation existed in practical engineering.

In this paper, we calculate the Normal Form in 1:2 strong internal resonance according to the definition of Poincaré resonance, and reduce the 4-dimension bifurcation equation to 3-dimension by using a technique of transformation, and obtain an order 4 bifurcation equation only with one variable. Then we analyze the bifurcation of universal unfolding and dynamic behavior based on the singularity theory. Under the case of 1 : 2 internal resonance, a high-dimension nonlinear dynamic system has two pairs of pure complex eigenvalues. It can be reduced to 4-dimension manifold by using the method of center manifold and analyzed the local bifurcation behavior with the method we suggest here.

1 The Normal Form in 1:2 Strong Internal Resonance

Here we consider nonlinear dynamic system governed by

$$y' = f(y) \qquad f \in C^k(R^n). \tag{1}$$

Through a series of nonlinear coordination transformation,

$$y = x + p_k(x) \qquad (k = 2, \cdots, l - 1). \tag{2}$$

Eq.(1) can be simplified to the normal form. Under the case of nonresonance, the Normal Form only contains linear terms, but when it satisfies the condition of resonance, nonlinearterms will be remained.

During the procedure of transformation, one needs to calculate an operator, the dimension of the operator becomes very huge as the order's highing, so the procedure of the classical method is very boring and difficult.

WU Zhi-qiang[4] supply a direct method to calculate the Normal Form. Owing to avoiding the calculation of ploy-matrix verse, combining the procedure of center manifold and Normal Form, the Normal Form will be obtained more easily.

Using this method, the dimension of equations required to solve will maintain invariant, just equate to the dimension of the original system, don't increase with the order of nonlinear terms enlarge.

The equations we study here come from a coupled system having multi-degrees of freedom, the nonlinear dynamic equations were derived by Lagrange method, reduced by using center manifold to 4-dimension equations[6]. A turbo-generating set, when we consider a symmetry flex-rotor, the system has four-degrees of freedom; the equations of this system have the same form such as Eq.(3) in 1:2 internal resonance.

Here we consider a coupled system having two degrees of freedom governed by:

$$\begin{cases} \ddot{x} + x + \delta\dot{x} + \varepsilon k_1 y + \varepsilon c_1 \dot{y} = xy + x\dot{x}y + \dot{x}^2 y + y^2 + x^2 y + x^3 = f_1, \\ \ddot{y} + (4+\mu)y + \delta c_2 \dot{y} + k_2 x + c_3 \dot{x} = \\ \qquad 2xy + x\dot{x}y + 2\dot{x}^2 y + 3y^2 + x^2 y + x^3 = f_2, \end{cases} \tag{3}$$

where x, y- displacement in two directions x and y of two bodies; c_1, c_2, c_3 and k_1, k_2 are coefficients of dampping and stiffness; ε, δ- small parameters. Eq.(3) contain quadratic and cubic nonlinearities at the same time.

To show how to obtain the Normal Form of the system in 1:2 internal resonance, we write Eq.(3) in the form of:

$$\dot{X} = AX + BX + F(x), \tag{4}$$

where

$$X = (x_1, x_2, x_3, x_4)^{\mathrm{T}}, \tag{5}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -k_2 & -c_3 & -4 & 0 \end{pmatrix}, \tag{6}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\delta & -\varepsilon k_1 & -\varepsilon c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & -\delta c_2 \end{pmatrix}, \tag{7}$$

$$F(x) = \begin{pmatrix} 0 \\ -f_1 \\ 0 \\ -f_2 \end{pmatrix}, \tag{8}$$

Letting

$$X = TZ + H(Z), \tag{9}$$

where

$$Z = (z_1, \bar{z}_1, z_2, \bar{z}_2)^{\mathrm{T}}, \tag{10}$$

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ I & 0 & -I & 0 \\ -\frac{1}{3}k_2 - \frac{1}{3}Ic_3 & 1 & -\frac{1}{3}k_2 + \frac{1}{3}Ic_3 & 1 \\ -\frac{1}{3}k_2 - \frac{1}{3}Ic_3 & 2I & \frac{1}{3}I(k_2 - Ic_3) & -2I \end{pmatrix}, \tag{11}$$

$$H(Z) = \sum_{|m| \geq 2} H_m Z^m. \tag{12}$$

Substituting (9) to (4), we obtain the following equations:

$$\dot{Z} = JZ + C(Z), \tag{13}$$

where

$$J = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 2I & 0 \\ 0 & 0 & 0 & -2I \end{pmatrix}, \tag{14}$$

$$C(Z) = \sum_{|m| \geq 2} C_m Z^m . \tag{15}$$

$H(Z)$ and $C(Z)$ satisfy the following relation:

$$[\langle m, \lambda \rangle J - A] H_m = \tilde{f}_m - TC_m . \tag{16}$$

According to the definition of Poincaré resonance,

$$\langle m, \lambda \rangle = \lambda_s . \tag{17}$$

Letting the maintain terms in form $z_1^{m_1} \bar{z}_1^{m_2} z_2^{m_3} \bar{z}_2^{m_4}$.

The case 1/2 internal resonance, one obtains

$$m_1 I - m_2 I + 2m_3 I - 2m_4 I = I, \tag{18}$$

$$m_1 I - m_2 I + 2m_3 I - 2m_4 I = -I, \tag{19}$$

$$m_1 I - m_2 I + 2m_3 I - 2m_4 I = 2I, \tag{20}$$

$$m_1 I - m_2 I + 2m_3 I - 2m_4 I = -2I. \tag{21}$$

Linear terms in Normal Form:

from (18): $m_1 = 1$ Retain z_1,

form (19): $m_2 = 1$ Retain $\bar{z}_1$,

form (20): $m_3 = 1$ Retain z_2,

form (21): $m_4 = 1$ Retain $\bar{z}_2$.

Quadratic terms in Normal Form:

from (18): $m_2 = 1$, $m_3 = 1$ Retain $\bar{z}_1 z_2$,

from (19): $m_1 = 1$, $m_4 = 1$ Retain $z_1 \bar{z}_2$,

form (20): $m_1 = 2$ Retain z_1^2,

form (21): $m_2 = 2$ Retain $\bar{z}_1^2$.

Cubic terms in Normal Form:

form (18): $m_1 = 2$, $m_2 = 1$; $m_1 = 1$, $m_3 = 1$, $m_4 = 1$
Retain $z_1^2 \bar{z}_1$, $z_1 z_2 \bar{z}_2$,

form (19): $m_1 = 1$, $m_2 = 2$; $m_2 = 1$, $m_3 = 1$, $m_4 = 1$
Retain $z_1 \bar{z}_1^2$, $\bar{z}_1 z_2 \bar{z}_2$,

form (20): $m_1 = 1$, $m_2 = 1$, $m_3 = 1$; $m_3 = 2$, $m_4 = 1$
Retain $z_1 \bar{z}_1 z_2$, $z_2^2 \bar{z}_2$,

form (21): $m_1 = 1$, $m_2 = 1$, $m_4 = 1$; $m_3 = 1$, $m_4 = 2$
Retain $z_1 \bar{z}_1 \bar{z}_2$, $z_2 \bar{z}_2^2$.

If the relative coefficients wrote as $c_{m_1 m_2 m_3 m_4}$ and $e_{m_1 m_2 m_3 m_4}$, and eliminate $\bar{z}_1$ and $\bar{z}_2$, the Normal

Form of this system in 1:2 internal resonance will be

$$\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix}=\begin{pmatrix}I & 0\\ 0 & 2I\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}c_{0110}\bar z_1 z_2\\ e_{2000}z_1^2\end{pmatrix}+\begin{pmatrix}c_{2100}z_1^2\bar z_1+c_{1011}z_1 z_2\bar z_2\\ e_{1110}z_1\bar z_1 z_2+e_{0021}z_2^2\bar z_2\end{pmatrix}, \tag{22}$$

c_{0110}, c_{2100}, c_{1011}, e_{2000}, e_{1110}, e_{0021} are complex related to original system.

Where

$$c_{0110}=\frac{8}{51}k_2-\frac{8}{51}Ic_3-\frac{1}{17}I+\frac{2}{51}Ik_2+\frac{2}{51}c_3-\frac{4}{17}, \tag{23}$$

$$c_{2100}=-\frac{1}{91\,800}I(549c_3k_2-1\,368Ik_2^2-54Ic_3^2-44\,649Ic_3+1\,080Ik_2+762Ik_2c_3-82\,890k_2-1\,080c_3+4\,527c_3^2+6\,054k_2^2+137\,700), \tag{24}$$

$$c_{1011}=\frac{1}{6\,120}I(729k_2+80c_3^2-153c_3+80k_2^2-2\,727+816Ic_3), \tag{25}$$

$$e_{2000}=\frac{13}{150}c_3k_2-\frac{1}{25}Ik_2^2+\frac{7}{150}Ic_3^2+\frac{2}{15}Ic_3+\frac{1}{15}Ik_2-\frac{7}{50}Ik_2c_3+\frac{2}{15}k_2-\frac{1}{15}c_3+\frac{3}{50}c_3^2-\frac{2}{25}k_2^2, \tag{26}$$

$$e_{1110}=\frac{1}{1\,101\,600}I(-5\,238c_3k_2+23\,481Ik_2^2-8\,352Ic_3^2-898\,290Ic_3-17\,010Ik_2+116\,271Ik_2c_3-45\,090k_2-12\,960c_3-6\,174c_3^2+31\,347k_2^2-1\,927\,800), \tag{27}$$

$$e_{0021}=-\frac{1}{20}I(k_2+6+2Ic_3). \tag{28}$$

2 Reduce 4-Dimension to 3-Dimension

Here the Normal Form in 1:2 internal resonance is in the form of

$$\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix}=\begin{pmatrix}I & 0\\ 0 & 2I\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}c_{0110}\bar z_1 z_2\\ e_{2000}z_1^2\end{pmatrix}+\begin{pmatrix}c_{2100}z_1^2\bar z_1+c_{1011}z_1 z_2\bar z_2\\ e_{1110}z_1\bar z_1 z_2+e_{0021}z_2^2\bar z_2\end{pmatrix}. \tag{22}$$

As the quadratic and cubic non-linearities exist in the Normal Form at the same time, the method of reduce we used before will face to many problem, it is impossible to reduce the equations from 4-dimension to 3-dimension, so the bifurcation can't be obtained. Now we suggest a new transformation

$$\rho=|z_1|^2,\ \frac{1}{\rho}(u+\mathrm{i}v)=\frac{z_2}{z_1^2}. \tag{29}$$

From (29)

$$z_1=\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta},\ z_2=(u+\mathrm{i}v)\mathrm{e}^{2\mathrm{i}\theta}, \tag{30}$$

$$\dot z_1=\frac{\dot\rho}{2\sqrt{\rho}}\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\dot\theta\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta},$$

$$\dot{z}_2 = (\dot{u} + i\dot{v})e^{2i\theta} + 2i\dot{\theta}(u + iv)e^{2i\theta}. \tag{31}$$

Substituting (30) and (31) to (22) and separating the result into real and imaginary parts, letting the real of c_{0110}, c_{2100}, c_{1011}, e_{2000}, e_{1110}, e_{0021} are a_i, the imaginary of c_{0110}, c_{2100}, c_{1011}, e_{2000}, e_{1110}, e_{0021} are b_i, where $i = 1,\cdots,6$, we obtain

$$\dot{\rho} = 2a_1\rho u - 2b_1\rho v + 2a_2\rho^2 + 2a_3(u^2 + v^2)\rho, \tag{32}$$

$$\dot{\theta} = 1 + b_1 u + a_1 v + b_2\rho + b_3(u^2 + v^2), \tag{33}$$

$$\dot{u} = 2v(\dot{\theta} - 1) + a_4\rho + a_5\rho u - b_5\rho v + a_6(u^2 + v^2)u - b_6(u^2 + v^2)v, \tag{34}$$

$$\dot{v} = -2u(\dot{\theta} - 1) + b_4\rho + a_5\rho v + b_5\rho u + a_6(u^2 + v^2)v + b_6(u^2 + v^2)u. \tag{35}$$

Substituting (33) to (34) and (35), we obtain

$$\dot{u} = 2b_1 uv + 2a_1 v^2 + 2b_2\rho v + 2b_3(u^2 + v^2)v + a_4\rho + a_5\rho u - b_5\rho v + a_6(u^2 + v^2)u - b_6(u^2 + v^2)v, \tag{36}$$

$$\dot{v} = -2b_1 u^2 - 2a_1 uv - 2b_2\rho u - 2b_3(u^2 + v^2)u + b_4\rho + a_5\rho v + b_5\rho u + a_6(u^2 + v^2)v + b_6(u^2 + v^2)u, \tag{37}$$

Here we show that how to gain the bifurcation equation.

Letting $\dot{\rho} = \dot{u} = \dot{v} = 0$, we obtain

$$2a_1\rho u - 2b_1\rho v + 2a_2\rho^2 + 2a_3(u^2 + v^2)\rho = 0, \tag{38}$$

$$2b_1 uv + 2a_1 v^2 + 2b_2\rho v + 2b_3(u^2 + v^2)v + a_4\rho + a_5\rho u - b_5\rho v + a_6(u^2 + v^2)u - b_6(u^2 + v^2)v = 0, \tag{39}$$

$$-2b_1 u^2 - 2a_1 uv - 2b_2\rho u - 2b_3(u^2 + v^2)u + b_4\rho + a_5\rho v + b_5\rho u + a_6(u^2 + v^2)v + b_6(u^2 + v^2)u = 0. \tag{40}$$

Eliminating u and v from (38) through (40) yields

$$A\rho^4 + B\rho^3 + C\rho^2 + D\rho + E = 0, \tag{41}$$

where A, B, C, D, E are coefficients related to original system.

3 Transition Sets and Bifurcation Diagrams

Eq. (41) is an algebraic equation with one variable four power. In general, it can be simplified by using a suitable transformation and eliminate the second power, so Eq. (41) can be written as follows:

$$x^4 + ax^2 + bx + c = 0. \tag{42}$$

Eq. (42) can be written as

$$x^4 + 2\eta\lambda x^2 + \lambda^2 + \alpha_1\lambda x + \alpha_2 x^2 + \alpha_3 x + \alpha_4 = 0, \tag{43}$$

Eq. (43) can be seen as a 4-parameter unfolding of following germ

$$g = x^4 + 2\eta\lambda x^2 + \lambda^2 \tag{44}$$

Bifurcation point set

B_1: $\alpha_4 = 0$,

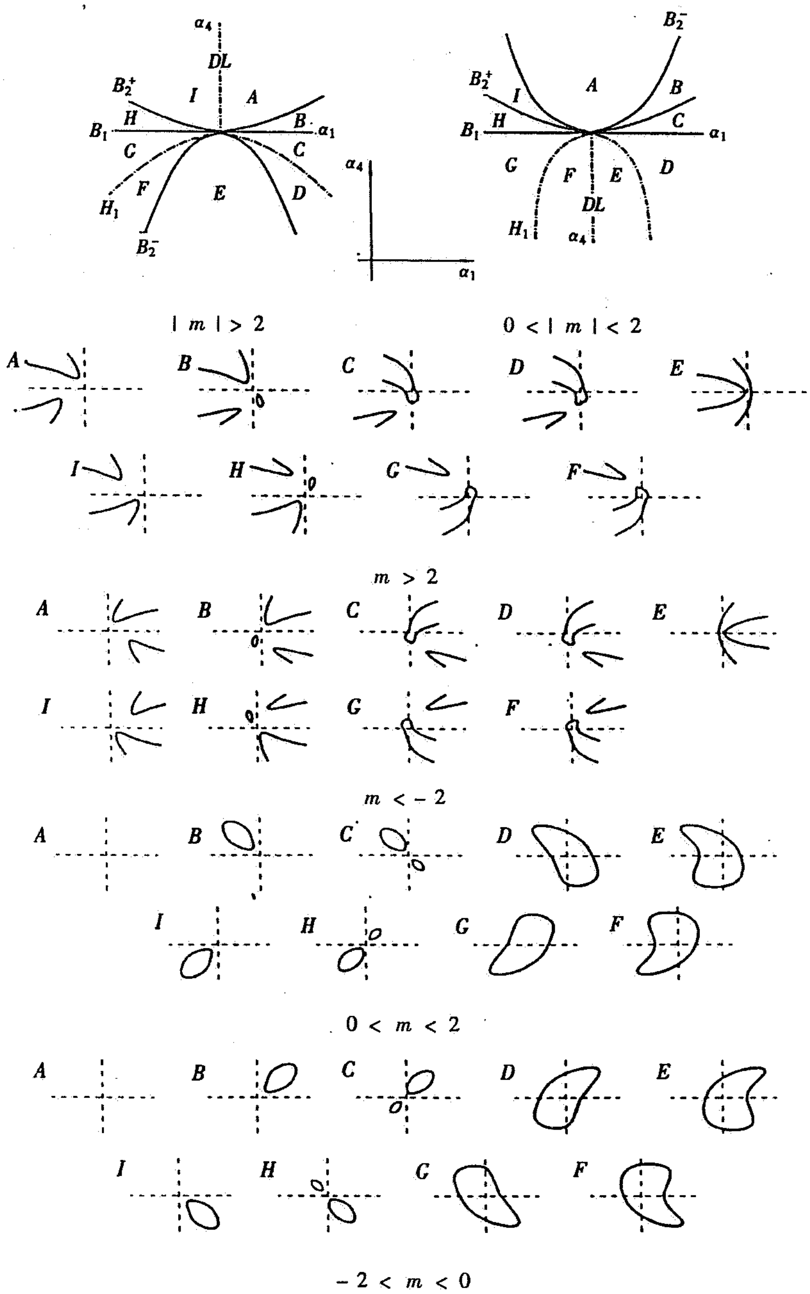
$$B_2^{\pm}: \quad \alpha_4 = \frac{1}{128}\frac{(m^4 - 80m^2 - 128 \pm \sqrt{m^2(m^2+32)^3})\alpha_1^4}{(m-2)^3(m+2)^3};$$

Hysteresis set

$$H: \quad \alpha_4 = -\frac{243}{256}\frac{(m^2+12)\alpha_1^4}{m^6};$$

Double limit point set

$$DL: \quad \alpha_1 = 0 \text{ and } \alpha_4(m^2-4) \geqslant 0.$$



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